General relativistic hydrodynamics and magnetohydrodynamics: hyperbolic systems in relativistic astrophysics

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1 Introduction

Einstein's theory of general relativity plays a major role in astrophysics, particularly in scenarios involving compact objects such as neutron stars and black holes. Those include gravitational collapse, γ-ray burts, accretion, relativistic jets in active galactic nuclei, or the coalescence of compact neutron star (or black hole) binaries. Astronomers have long been scrutinizing these systems using the complete frequency range of the electromagnetic spectrum. Nowadays, they are the main targets for ground-based laser interferometers of gravitational radiation. The direct detection of these elusive ripples in the curvature of spacetime, and the wealth of new information that could be extracted thereof, is one of the driving motivations of present-day research in relativistic astrophysics.

Theoretical astrophysics has long relied on numerical simulations as a formidable way to improve our understanding of the dynamics of astrophysical systems. For the case we are concerned with in this paper, the mathematical framework upon which such simulations are based is nowadays developed to high levels of sophistication. The equations governing the dynamics of relativistic astrophysical systems are an intricate set of coupled, time-dependent partial differential equations, comprising the general relativistic hydrodynamics and magnetohydrodynamics equations (GRHD/GRMHD hereafter) and Einstein's gravitational field equations. Simplifications can be made when the "test-fluid" approximation holds, in which the fluid's self-gravity is neglected against the background gravitational field. Additionally, descriptions employing ideal hydrodynamics (inviscid fluids) and ideal MHD (infinite conductivity), are also fairly standard choices in numerical astrophysics. On the other hand, there are also situations where the number of equations must be augmented instead, as to account for radiative processes or microphysics (finite-temperature equations of state (EOS) and nuclear physics).

This article aims at presenting a brief overview of the equations of GRHD/GRMHD within the 3+1 formalism of general relativity in ways suitable for numerical work. Space constraints limit the level of detail of the presentation, which the interested reader can complement with the help of the available literature on the subject (see [**?**, **?**, **?**, **?**] and references therein). The article also discusses the different numerical approaches designed to solve hyperbolic systems of conservation laws such as the GRHD/GRMHD equations. Some examples in the numerical solution in scenarios of relativistic astrophysics are mentioned in the last section.

2 General relativistic hydrodynamics

The GRHD equations are the local conservation laws of momentum and energy, encoded in the stress-energy tensor $T^{\mu\nu}$, and of the matter density, J^μ (the continuity equation)

$$\nabla_\mu T^{\mu\nu} = 0, \qquad \nabla_\mu J^\mu = 0, \tag{1}$$

where ∇_μ stands for the 4-dimensional covariant derivative. (Throughout Greek indices run from 0 to 3 and Latin indices from 1 to 3; geometrized units $G = c = 1$ are used; G is Newton's gravitational constant and c is the speed of light.) The density current reads $J^\mu = \rho u^\mu$, where u^μ is the 4-velocity of the fluid and ρ its rest-mass density. We assume a perfect fluid stress-energy tensor

$$T^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu}, \tag{2}$$

where p is the pressure, h is the specific enthalpy, $h = 1 + \varepsilon + p/\rho$, ε being the specific internal energy, and $g_{\mu\nu}$ is the spacetime metric tensor.

The previous system of equations is closed once a EOS is chosen, i.e. a constitutive relation of the form $p = p(\rho, \varepsilon)$. In the so-called test-fluid approximation the dynamics of the matter fields is completely described by the previous conservation laws and the EOS. If such approximation does not hold, these equations must be solved in conjunction with Einstein's equations for the gravitational field which describe the evolution of a dynamical spacetime.

The approach most commonly employed to solve Einstein's equations in Numerical Relativity is the so-called Cauchy or 3+1 formulation (IVP) (see [**?**] and references therein for details). In this formulation spacetime is foliated into a set of non-intersecting spacelike hypersurfaces, for which the lapse function α measures the rate of advance of time along a timelike unit vector n^μ normal to a surface, and the spacelike shift vector β^i describes how coordinates move between different hypersurfaces. Introducing a coordinate chart (x^0, x^i) the 3+1 line element reads

$$ds^2 = -(\alpha^2 - \beta_i\beta^i)dx^0dx^0 + 2\beta_i dx^i dx^0 + \gamma_{ij}dx^i dx^j, \tag{3}$$

where γ_{ij} is the spatial 3-metric induced on each spacelike slice. Hence, the above conservation equations, Eq. (**??**), read

$$\frac{\partial}{\partial x^\mu}\sqrt{-g}J^\mu = 0, \qquad \frac{\partial}{\partial x^\mu}\sqrt{-g}T^{\mu\nu} = -\sqrt{-g}\Gamma^\nu_{\mu\lambda}T^{\mu\lambda}\,, \tag{4}$$

where $g = \det(g_{\mu\nu}) = \alpha\sqrt{\gamma}$, $\gamma = \det(\gamma_{ij})$ and $\Gamma^\nu_{\mu\lambda}$ are the so-called Christoffel symbols.

Chronologically, the first attempt to formulate and solve the equations of relativistic Eulerian hydrodynamics in multidimensions was due to Wilson [**?**], who wrote the system as a coupled set of advection equations within the 3+1 formalism. This approach sidestepped an important guideline for the formulation of nonlinear hyperbolic equations, namely the preservation of their conservation form. This is a necessary feature to guarantee correct evolution in regions of entropy generation (shocks). As a result, some amount of numerical dissipation (artificial viscosity terms) had to be used to stabilize the numerical solution at discontinuities.

The main practical limitation of such non-conservative system was the numerical inability to handle ultrarelativistic flows [**?**]. This handicap posed a tremendous challenge to the numerical modelling of relativistic astrophysical sources where flow velocities as large as 99% of the speed of light or higher are known to exist. Paradigmatic examples of such sources are the jets associated with active galactic nuclei as well as with γ-ray bursts, the most luminous events in the Universe after the Big Bang. On the other hand, the presence of the Lorentz factor ($W \equiv \alpha u^0$) in the convective (transport) terms of the GRHD equations (and of the pressure in the specific enthalpy) make the relativistic equations much more coupled than their Newtonian counterparts. In an attempt to capture more accurately such coupling [**?**] proposed the use of implicit schemes. While some progress was achieved limitations on the fluid speeds attainable persisted, with a maximum value for W of about 10. Ultrarelativistic flows could only be handled (and with explicit schemes) once conservative formulations were adopted.

Since the early 1990s conservative formulations of the GRHD equations, well-adapted to numerical methodology, were developed: [**?**] (1+1, general EOS), [**?**] (covariant, perfect fluid EOS), [**?**] (3+1, general EOS), and [**?**] (covariant, general EOS). Numerically, the hyperbolic and conservative nature of the GRHD equations allows to design a solution procedure based on the characteristic speeds and fields of the system (i.e. Riemann solvers), translating to relativistic hydrodynamics existing tools of computational fluid dynamics. This procedure departs from earlier approaches [**?**], most notably in avoiding the need for artificial dissipation terms to handle discontinuous solutions as well as implicit schemes as proposed by [**?**].

The extension of modern *high-resolution shock-capturing* schemes (HRSC hereafter) from classical fluid dynamics to relativistic hydrodynamics was accomplished in three steps: a) Casting the GRHD equations as a system of conservation laws; b) identifying the suitable vector of unknowns; and c) building up an approximate Riemann solver. For brevity we focus next on the approach taken by [?]. The interested reader is addressed to the previous references for specific details on additional formulations.

In [?] the GRHD equations were written as a first-order, flux-conservative hyperbolic system, amenable to numerical work:

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\mathbf{U}(\mathbf{w})}{\partial x^0}+\frac{\partial\sqrt{-g}\mathbf{F}^i(\mathbf{w})}{\partial x^i}\right)=\mathbf{S}(\mathbf{w}). \tag{5}$$

With respect to an Eulerian observer and in terms of the *primitive variables*, $\mathbf{w}=(\rho,v^i,\varepsilon)$, where v^i is the 3-velocity of the fluid, the state vector $\mathbf{U}$ (conserved variables) and the vectors of fluxes $\mathbf{F}$ and source terms $\mathbf{S}$ are given by

$$\mathbf{U}(\mathbf{w})=(D,S_j,\tau), \tag{6}$$

$$\mathbf{F}^i(\mathbf{w})=\left(D\tilde{v}^i,S_j\tilde{v}^i+p\delta^i_j,\tau\tilde{v}^i+pv^i\right), \tag{7}$$

$$\mathbf{S}(\mathbf{w})=\left(0,T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu}-\Gamma^\delta_{\nu\mu}g_{\delta j}\right),\alpha\left(T^{\mu 0}\frac{\partial\ln\alpha}{\partial x^\mu}-T^{\mu\nu}\Gamma^0_{\nu\mu}\right)\right), \tag{8}$$

with $\tilde{v}^i=v^i-\beta^i/\alpha$ and δ^i_j being the Kronecker delta. The conserved quantities are the relativistic densities of mass, momenta, and energy, defined as $D=\rho W$, $S_j=\rho hW^2v_j$, and $\tau=\rho hW^2-p-D$.

HRSC schemes based on Riemann solvers use the local characteristic structure of the hyperbolic system of equations. The wave structure of the GRHD equations (**??**) analyzed in [?] (see also [?]) showed that the eigenvalues (characteristic speeds) of the corresponding Jacobian matrices are real and there exists a complete set of right-eigenvectors. System (**??**) satisfies, thus, the definition of hyperbolicity.

It is worth mentioning a key difference of the wave structure of the GRHD equations with respect to the Newtonian case. This is also apparent in the absence of gravity, that is, in special relativity, $g_{\mu\nu}$=diag$(-1,1,1,1)$, which we will adopt next for simplicity. In this case the eigenvalues (along the x-direction) read [?]

$$\lambda_0=v^x \qquad \text{(triple)} \tag{9}$$

$$\lambda_\pm=\frac{1}{1-v^2c_s^2}\left(v^x(1-c_s^2)\pm c_s\sqrt{(1-v^2)[1-v^xv^x-(v^2-v^xv^x)c_s^2]}\right) \tag{10}$$

where c_s is the speed of sound and $v^2=v^iv_i$. Thus, the eigenvalues along the x-direction involve the coupling of the components of the velocity transverse to the chosen direction (and similarly for the other two directions). Even in the purely one-dimensional case, $\mathbf{v}=(v^x,0,0)$, the eigenvalues read $\lambda_0=v^x$ and $\lambda_\pm=\frac{v^x\pm c_s}{1\pm v^xc_s}$, the latter involving a Lorentz addition of the fluid velocity and the sound speed, as opposed to the Galilean addition of the velocities appearing in the Newtonian case ($\lambda_\pm=v^x\pm c_s$).

A distinctive feature of the numerical solution of the RHD equations is that while the numerical algorithm updates the vector of conserved quantities $\mathbf{U}$ (see below), the numerical code makes extensive use of the primitive variables $\mathbf{w}$. Those would appear repeatedly in the solution procedure, e.g. in the characteristic fields, in the solution of the Riemann problem, and in the computation of the numerical fluxes. For spacelike foliations of the spacetime (i.e. 3+1) the relation between the two sets of variables turns out to be implicit. Therefore, iterative (root-finding) algorithms are required to recover the primitive variables. Those have been developed for all existing formulations [**?**, **?**, **?**]. This feature, absent in Newtonian hydrodynamics, may lead to accuracy losses in regions of low density and small velocities, apart from being computationally inefficient. Only for *null* foliations of the spacetime, the procedure of connecting primitive and conserved variables is explicit for a perfect fluid EOS, a direct consequence of the particular form of the Bondi-Sachs metric [**?**].

3 General relativistic magnetohydrodynamics

General relativistic MHD is concerned with the dynamics of relativistic, electrically conducting fluids (plasma) in the presence of magnetic fields. Here, we concentrate on purely *ideal* GRMHD, neglecting the presence of viscosity and heat conduction in the limit of infinite conductivity (perfect conductor fluid). As the GRHD equations discussed before, the GRMHD equations can also be cast in first-order, flux-conservative hyperbolic form. The discussion reported here follows the derivation of these equations as presented in [**?**] to which the reader is addressed for details (see also references therein).

In terms of the (Faraday) electromagnetic tensor $F^{\mu\nu}$, Maxwell's equations read

$$\nabla_\nu {}^*F^{\mu\nu} = 0, \qquad \nabla_\nu F^{\mu\nu} = \mathcal{J}^\mu, \tag{11}$$

where $F^{\mu\nu} = U^\mu E^\nu - U^\nu E^\mu - \eta^{\mu\nu\lambda\delta} U_\lambda B_\delta$, its dual ${}^*F^{\mu\nu} = \frac{1}{2}\eta^{\mu\nu\lambda\delta} F_{\lambda\delta}$, and $\eta^{\mu\nu\lambda\delta} = \frac{1}{\sqrt{-g}}[\mu\nu\lambda\delta]$, where $[\mu\nu\lambda\delta]$ is the completely antisymmetric Levi-Civita symbol. E^μ and B^μ stand for the electric and magnetic fields measured by an observer with 4-velocity U^μ, and $\mathcal{J}^\mu$ is the electric 4-current, $\mathcal{J}^\mu = \rho_q u^\mu + \sigma F^{\mu\nu} u_\nu$ where ρ_q is the proper charge density and σ is the electric conductivity.

Maxwell's equations can be simplified if the fluid is a perfect conductor. In this case σ is infinite and, to keep the current finite, the term $F^{\mu\nu} u_\nu$ must vanish, which results in $E^\mu = 0$ for a comoving observer. This case corresponds to the so-called ideal MHD condition. Under this assumption the electric field measured by the Eulerian observer has components

$$E^0 = 0, \qquad E^i = -\alpha \eta^{0ijk} v_j B_k, \tag{12}$$

and Maxwell's equations $\nabla_\nu {}^*F^{\mu\nu} = 0$ reduce to the divergence-free condition plus the induction equation for the evolution of the magnetic field

$$\frac{\partial(\sqrt{\gamma}B^i)}{\partial x^i} = 0, \qquad \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^0}(\sqrt{\gamma}B^i) = \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^j}\{\sqrt{\gamma}[\alpha\tilde{v}^i B^j - \alpha\tilde{v}^j B^i]\}. \quad (13)$$

For a fluid endowed with a magnetic field the stress-energy tensor is the sum of that of the fluid and that of the electromagnetic field, $T^{\mu\nu} = T^{\mu\nu}_{\rm Fluid} + T^{\mu\nu}_{\rm EM}$, where $T^{\mu\nu}_{\rm Fluid}$ is given by Eq. (**??**) for a perfect fluid. On the other hand $T^{\mu\nu}_{\rm EM}$ can be obtained from the Faraday tensor as follows:

$$T^{\mu\nu}_{\rm EM} = F^{\mu\lambda}F^{\nu}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\delta}F_{\lambda\delta}, \quad (14)$$

which, in ideal MHD, can be rewritten as

$$T^{\mu\nu}_{\rm EM} = \left(u^\mu u^\nu + \frac{1}{2}g^{\mu\nu}\right)b^2 - b^\mu b^\nu, \quad (15)$$

where b^μ is the magnetic field measured by the observer comoving with the fluid and $b^2 = b^\nu b_\nu$. The total stress-energy tensor is thus given by

$$T^{\mu\nu} = \rho h^* u^\mu u^\nu + p^* g^{\mu\nu} - b^\mu b^\nu, \quad (16)$$

with the definitions $p^* = p + b^2/2$ and $h^* = h + b^2/\rho$.

Following [**?**] the conservation equations for the energy-momentum tensor, together with the continuity equation and the equation for the evolution of the magnetic field, can be written as a first-order, flux-conservative, hyperbolic system equivalent to (**??**). The state vector and the vector of fluxes of the GRMHD system of equations read:

$$\mathbf{U}(\mathbf{w}) = (D, S_j, \tau, B^k), \quad (17)$$

$$\mathbf{F}^i(\mathbf{w}) = (D\tilde{v}^i, S_j\tilde{v}^i + p^*\delta^i_j - b_j B^i/W, \tau\tilde{v}^i + p^* v^i - \alpha b^0 B^i/W, \tilde{v}^i B^k - \tilde{v}^k B^i), \quad (18)$$

where the conserved quantities are now defined as $D = \rho W$, $S_j = \rho h^* W^2 v_j - \alpha b^0 b_j$, and $\tau = \rho h^* W^2 - p^* - \alpha^2(b^0)^2 - D$. The corresponding vector of sources coincides with the one given by Eq. (**??**) save for the use of the complete (fluid plus electromagnetic field) stress-energy tensor (the magnetic field evolution equation is source-free).

The hyperbolic structure of those equations is discussed in [**?**]. In the classical MHD case the wave structure was analyzed by [**?**]. There are seven physical waves: two Alfvén waves (with eigenvalues $\lambda_{a\pm} = v_x \pm v_a$, v_x and v_a being the fluid and Alfvén speeds, respectively), two fast and two slow magnetosonic waves ($\lambda_{f\pm} = v_x \pm v_f$, $\lambda_{s\pm} = v_x \pm v_s$), and one entropy wave ($\lambda_e = v_x$), ordered such that $\lambda_{f_-} < \lambda_{a_-} < \lambda_{s_-} < \lambda_e < \lambda_{s_+} < \lambda_{a_+} < \lambda_{f_+}$. The expressions for the Alfvén and magnetosonic speeds read

$$v_a = \sqrt{\frac{B_x^2}{\rho}}, \tag{19}$$

$$v_{f,s}^2 = \frac{1}{2}\left\{c_s^2 + \frac{B_x^2+B_y^2+B_z^2}{\rho} \pm \sqrt{\left(c_s^2 + \frac{B_x^2+B_y^2+B_z^2}{\rho}\right)^2 - 4v_a^2c_s^2}\right\} \tag{20}$$

The corresponding wave structure for relativistic MHD was thoroughly analyzed by [?]. The investigation of the roots of the characteristic equation showed that only the entropic waves and the Alfvén waves can be (explicitely) obtained in closed form, while the magnetosonic waves are given by the numerical solution of a quartic equation.

For the GRMHD formulation of [?] the characteristic speed of the entropic waves propagating in the x-direction reads

$$\lambda_e = \alpha v^x - \beta^x. \tag{21}$$

For Alfvén waves, there are two solutions corresponding to different speeds of the waves,

$$\lambda_{a\pm} = \frac{b^x \pm \sqrt{\rho h + b^2}u^x}{b^0 \pm \sqrt{\rho h + b^2}u^0}. \tag{22}$$

Just as in the classical case, the relativistic MHD equations have degenerate states in which two or more wavespeeds coincide, which breaks the strict hyperbolicity of the system. [?] has reviewed the properties of these degeneracies. In the fluid rest frame, the degeneracies in both classical and relativistic MHD are the same: either the slow and Alfvén waves have the same speed as the entropy wave when propagating perpendicularly to the magnetic field (Degeneracy I), or the slow or the fast wave (or both) have the same speed as the Alfvén wave when propagating in a direction aligned with the magnetic field (Degeneracy II). These degeneracies have been characterized by [?] in terms of the components of the magnetic field 4-vector normal and tangential to the Alfvén wavefront, $\mathbf{b}_n$, $\mathbf{b}_t$. When $\mathbf{b}_n = 0$, the system falls within Degeneracy I, while Degeneracy II is reached when $\mathbf{b}_t = 0$. In addition, [?] have also worked out a single set of right and left eigenvectors which are regular and span a complete basis in any physical state, including degenerate states. Such renormalization procedure is a relativistic generalization of the work performed by [?] in classical MHD.

On the other hand, as for the case of the GRHD equations discussed before, iterative (root-finding) algorithms are also required for the GRMHD equations to recover the primitive variables from the state vector. The recovery procedure is in this case more involved than for unmagnetized flows. For the GRMHD formulation discussed in this section [?] find the roots of an 8-th order polynomial using a two-dimensional Newton-Raphson scheme. The interested reader is addressed to [?] for a comparison of different methods.

We end this section by pointing out that major advances on the physical understanding of the wave structure of the relativistic hydrodynamics equations have been possible in recent years, remarkably thanks to the derivation of exact solutions of the Riemann problem both in special relativistic hydrodynamics and MHD [**?**, **?**, **?**, **?**, **?**].

4 Solution procedure for the GRHD/GRMHD equations

Just as their Newtonian counterparts, the GRHD/GRMHD equations are nonlinear hyperbolic systems of conservation laws. A distinctive feature of such systems is that smooth initial data can develop discontinuities during the time evolution. It is well known that standard finite difference schemes show deficiencies when dealing with discontinuous solutions. Typically, first order accurate schemes are too dissipative across discontinuities while second order (or higher) schemes produce spurious oscillations near discontinuities.

Finite difference schemes provide numerical solutions of the discretised version of the partial differential equations (PDEs). Therefore, convergence properties under grid refinement must be enforced on such schemes to guarantee the validity of the numerical result. The Lax-Wendroff theorem states that for hyperbolic systems of conservation laws, schemes written in *conservation form* converge to one of the so-called *weak solutions* of the PDEs. However, the class of all weak solutions is too wide as there is no uniqueness for the IVP. Thus, among all weak solutions, the numerical scheme must guarantee convergence to the *physically admissible solution*, a property whose mathematical characterisation was given by Lax for hyperbolic systems of conservation laws.

A conservative scheme for system (**??**) can be straightforwardly devised by using the corresponding integral form:

$$\int_{\Omega} \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{\gamma}\mathbf{U}}{\partial x^0} d\Omega + \int_{\Omega} \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}\mathbf{F}^i}{\partial x^i} d\Omega = \int_{\Omega} \mathbf{S} d\Omega, \tag{23}$$

where Ω is a region of the 4-dimensional manifold enclosed within a 3-dimensional surface $\partial\Omega$ which is bounded by two spacelike surfaces $\Sigma_{x^0}, \Sigma_{x^0+\Delta x^0}$ and two timelike surfaces $\Sigma_{x^i}, \Sigma_{x^i+\Delta x^i}$. For numerical purposes the above relation can be written as:

$$\begin{aligned}
\bar{\mathbf{U}}_{t+\Delta t} - \bar{\mathbf{U}}_t = & -\left(\int_{\Sigma_{x^1+\Delta x^1}} \sqrt{-g}\hat{\mathbf{F}}^1 dx^0 dx^2 dx^3 - \int_{\Sigma_{x^1}} \sqrt{-g}\hat{\mathbf{F}}^1 dx^0 dx^2 dx^3 \right) \\
& -\left(\int_{\Sigma_{x^2+\Delta x^2}} \sqrt{-g}\hat{\mathbf{F}}^2 dx^0 dx^1 dx^3 - \int_{\Sigma_{x^2}} \sqrt{-g}\hat{\mathbf{F}}^2 dx^0 dx^1 dx^3 \right) \\
& -\left(\int_{\Sigma_{x^3+\Delta x^3}} \sqrt{-g}\hat{\mathbf{F}}^3 dx^0 dx^1 dx^2 - \int_{\Sigma_{x^3}} \sqrt{-g}\hat{\mathbf{F}}^3 dx^0 dx^1 dx^2 \right) \\
& + \int_{\Omega} \mathbf{S} d\Omega,
\end{aligned} \tag{24}$$

where

$$\bar{\mathbf{U}} = \frac{1}{\Delta V} \int_{x^1}^{x^1+\Delta x^1} \int_{x^2}^{x^2+\Delta x^2} \int_{x^3}^{x^3+\Delta x^3} \sqrt{\gamma}\mathbf{U} dx^1 dx^2 dx^3 \tag{25}$$

and

$$\Delta V = \int_{x^1}^{x^1+\Delta x^1} \int_{x^2}^{x^2+\Delta x^2} \int_{x^3}^{x^3+\Delta x^3} \sqrt{\gamma} dx^1 dx^2 dx^3. \tag{26}$$

The main advantage of this procedure is that those variables which obey a conservation law are conserved during the evolution, as long as the balance between the fluxes at the boundaries of the computational domain and the source terms are zero. The *numerical fluxes* appearing in Eq. (**??**) are calculated at cell interfaces where the flow conditions can be discontinuous. Those numerical fluxes are approximations to the time-averaged fluxes across an interface, i.e.

$$\hat{\mathbf{F}}_{i+\frac{1}{2}} = \frac{1}{\Delta x^0} \int_{x^{0\,n}}^{x^{0\,n+1}} \mathbf{F}(\mathbf{U}(x_{i+\frac{1}{2}}, x^0)) dx^0, \tag{27}$$

where the flux integral depends on the solution at the numerical interfaces, $\mathbf{U}(x_{i+1/2}, x^0)$, during a time step. Godunov first proposed to calculate $\mathbf{U}(x_{i+1/2}, x^0)$ by exactly solving Riemann problems at every cell interface to obtain $\mathbf{U}(x_{i+1/2}, x^0) = \mathbf{U}(0; \mathbf{U}_i^n, \mathbf{U}_{i+1}^n)$, which denotes the Riemann solution for the (left and right) states $\mathbf{U}_i^n$, $\mathbf{U}_{i+1}^n$ along the ray $x^i/x^0 = 0$. This was a procedure of far-reaching consequences as it was incorporated in the design of numerical schemes for solving the Euler equations of classical gas dynamics in the presence of shock waves, which led to major advances in the field.

The derivation of the exact Riemann solution involves the computation of the full wave speeds to find where they lie in state space. This is a computationally expensive procedure, particularly for complex EOS and in multidimensions. Furthermore, for relativistic multidimensional flows, the coupling of all velocity components through the Lorentz factor results in the increase in the number of algebraic Rankine-Hugoniot conditions to consider in the case of shock waves and in solving a system of ODEs for the rarefaction waves. In spite of this the exact solution of the Riemann problem in *special* relativistic hydrodynamics has been derived [**?**, **?**]. Nevertheless, the computational inefficiency involved in the use of the exact solver in long-term numerical simulations motivated the gradual development of *approximate Riemann solvers*. These, being much cheaper than the exact solver yield equally accurate results.

The spatial accuracy of the numerical solution can be increased by reconstructing the primitive variables at the cell interfaces before the actual computation of the numerical fluxes. Diverse cell-reconstruction procedures are available in the literature (see references in [**?**, **?**]) and have been straightforwardly applied in relativistic hydrodynamics. Correspondingly, the temporal accuracy of the scheme can be improved by advancing in time the equations in integral form using the method of lines in tandem with a high-order, conservative Runge-Kutta method.

The main approaches extended from computational fluid dynamics to build HRSC schemes in relativistic hydrodynamics can be divided in the following broad categories:

1. HRSC schemes based on **Riemann solvers** (upwind methods): Developments include both solvers relying on the **exact solution** of the Riemann problem: [?, ?], relativistic PPM [?], Glimm's random choice method [?], and two-shock approximation [?, ?], as well as **linearized solvers** based on local linearizations of the Jacobian matrices of the flux-vector Jacobians, e.g. Roe-type Riemann solvers (Roe-Eulderink: [?]; Local Characteristic Approach: [?, ?, ?]), primitive-variable formulation: [?], and Marquina Flux Formula: [?].
2. HRSC schemes sidestepping the use of characteristic information (**symmetric schemes** with nonlinear numerical dissipation): Various approaches have been undertaken recently, including those by [?] (Lax-Wendroff scheme with conservative TVD dissipation terms), [?] (Lax-Friedrichs or HLL schemes with third-order ENO reconstruction algorithms), [?] (non-oscillatory central differencing), and [?, ?] (semidiscrete central scheme of Kurganov-Tadmor [?]).

Other approaches worth mentioning include: 1) artificial viscosity [?, ?], 2) flux-corrected transport scheme [?], and 3) smoothed particle hydrodynamics [?, ?]. The interested reader is addressed to the review article by [?] for a complete list of references on this topic as well as for an in-depth comparison of the performance of these various approaches.

The numerical advantage of using Eq. (??) for the hydrodynamical variables is not apparent for the magnetic field components, as there is no guarantee that such procedure conserves the divergence of the magnetic field during an evolution. The main physical implication of the divergence constraint is that the magnetic flux through a closed surface is zero. This property is essential to the design of the so-called constrained transport method [?, ?], a common choice among the methods designed to solve the induction equation while preserving the divergence of the magnetic field [?].

The current approaches to solve the RMHD equations within HRSC schemes also fall within the categories mentioned above, yet the development is somewhat more limited here than in the purely hydrodynamical case. Methods based on Riemann solvers have been initiated in special relativity by [?] (includes eigenvector sets for degenerate states), [?] (reconstruction not done on primitive variables), [?] (right and left eigenvectors in covariant variables, but 1D), and [?] (right and left eigenvectors in conserved variables, complete set even for degenerate states), as well as in general relativity by [?, ?]. On the other hand symmetric schemes (namely HLL and Kurganov-Tadmor) are being currently employed by a growing number of groups in GRMHD [?, ?, ?, ?, ?]. All references listed here use conservative formulations of the RMHD equations. Artificial viscosity approaches are advocated by [?, ?].

5 Applications in relativistic astrophysics

Numerical HD/MHD simulations are an essential tool in theoretical astrophysics, both to model classical and relativistic sources. In the latter case the progress achieved during the last few decades as a result of ever-increasing computational improvements as well as greater understanding of the mathematical aspects of the equations and of the numerical schemes to solve them, has been outstanding. Furthermore, its scope involves a large number of scenarios at the forefront of research in astrophysics which has only been possible to start approaching in recent times.

Examples include heavy ion collisions (in the special relativistic limit), formation and propagation of jets associated with both active galactic nuclei and γ-ray burst progenitors, gravitational stellar collapse to neutron stars and black holes, pulsations and instabilities of rotating relativistic stars, accretion on to black holes, and binary neutron star mergers. Such a list of applications is too large to allow for an adequate coverage within the space constraints of this article. Hence, only a paradigmatic example will be briefly discussed here, namely gravitational stellar core collapse to a neutron star, addressing the interested reader to [**?**, **?**] and references there in for more extended discussions.

The gravitational collapse of massive stars is a distinctive example in relativistic astrophysics involving self-gravitating fluids whose dynamics is governed by the GRHD/GRMHD equations coupled to Einstein's gravitational field equations. Stars with initial masses larger than $\sim 9M_{\odot}$ (where $M_{\odot}$ is the mass of the Sun) end their thermonuclear evolution developing a core composed of iron group nuclei, which is dynamically unstable against gravitational collapse. The core collapses to a neutron star releasing gravitational binding energy of the order $\sim 3\times 10^{53}$ erg $(M/M_{\odot})^2(R/10\ \mathrm{km})^{-1}$, sufficient to power a supernova explosion. Numerical simulations show how sensible the explosion mechanism is to the details of the post-bounce evolution: gravity, the nuclear EOS and the properties of the nascent neutron star, the treatment of the neutrino transport, and the neutrino-matter interaction. Only recently simulations including state-of-the-art neutrino transport, in which the Boltzmann equation is solved in connection with the hydrodynamics, are becoming possible (see [**?**] and references therein). Relativistic simulations of microphysically detailed core collapse beyond spherical symmetry are not yet available.

Steps towards that final goal are however being taken. Numerical simulations of (axisymmetric) relativistic rotational core collapse, approximating Einstein's equations for a conformally-flat 3-metric, were first reported in [**?**]. These did not include the necessary microphysics involved in supernova modelling as they aimed at computing the gravitational radiation from core collapse, to highlight the differences in the dynamics and waveforms between Newtonian and relativistic gravity. The gravitational wave signal is characterised by a burst associated with the hydrodynamical bounce followed by the proto-neutron star ringdown phase. While the central density reaches higher values in relativistic gravity the gravitational wave signals are of comparable amplitudes, which constraints the chances for detection to galactic events. Further simulations have improved the approximation in the metric equations [?] or use the full Einstein equations [?]. First attempts towards simulating GRMHD core collapse are currently being taken.

6 Summary

Formulations of the equations of (inviscid) general relativistic hydrodynamics and (ideal) magnetohydrodynamics have been discussed, along with methods for their numerical solution. Upon the explicit choice of an Eulerian observer and suitable fluid and magnetic field variables, it is possible to cast both systems of equations as first-order, hyperbolic systems of conservation laws. During the last fifteen years, the so-called (upwind) high-resolution shock-capturing schemes based on Riemann solvers have been extended from classical to relativistic fluid dynamics (both special and general), to the point that GRHD simulations in relativistic astrophysics are routinely performed nowadays. While such advances also hold true in the case of the MHD equations, the development still awaits here for a thorough numerical exploration. The article has also presented a brief overview of numerical techniques, providing examples of their applicability to general relativistic fluids and magneto-fluids in scenarios of relativistic astrophysics.

It is worth spending a last comment to mention the long-term, numerically stable formulations of Einstein's equations (or accurate enough approximations) that have been proposed by several Numerical Relativity groups worldwide in recent years. The paradigm which the numerical relativist is currently confronted with has suddenly changed for the best. Accurate and long-term stable, *coupled* evolutions of the GRHD/GRMHD equations and Einstein's equations are just becoming possible in three-dimensions (benefited from the steady increase in computing power), allowing for the study of interesting relativistic astrophysics scenarios for the first time, such as gravitational collapse, accretion onto black holes, and binary neutron star mergers.

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